Elementary maths for GMT

Calculus

Part 2.1: Integrals

The primitive

- A *primitive* function $F(x)$ of a function $f(x)$ is defined by

$$F'(x) = f(x)$$

- Any function $F(x)$ whose derivative equals $f(x)$ is a primitive of $f(x)$
- The primitive is often called the *antiderivative*, because you find a primitive by ‘inverting’ differentiation

Example

$$F'(x) = f(x)$$

primitive

x^2	is a primitive of	$2x$
$x^2 + 4$	is a primitive of	$2x$
$\sin x$	is a primitive of	$\cos x$

The primitive

- If F is a primitive of f, then every function
  $G(x) = F(x)+C$
  is also a primitive of f for every constant C, because $G' = F' = f$

The integral

- The collection of all primitives of $f(x)$ is called the **indefinite integral** and denoted as

$$\int f(x)dx$$

- So

$$\int f(x)dx = F(x) + C$$

Examples

- The indefinite integral can be found by 'inverting' differentiation

$$\int 2x \, dx = x^2 + C$$

$$\int x^3 \, dx = \tfrac{1}{4} x^4 + C$$

$$\int \cos x \, dx = \sin x + C$$

Properties

- The integral is a *linear* operation

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int \left(f(x)+g(x)\right)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

- Example

$$\int (2\cos 3x + 5\sin 4x)\,dx =$$

$$2\int \cos 3x\,dx + 5\int \sin 4x\,dx$$

Differential form

- The problem $\dfrac{dy}{dx} = f(x)$ for unknown $y=y(x)$ can often be solved by taking the integral of both sides:

$$y(x) = \int f(x)\, dx + C$$

Example

- Solve $\frac{dy}{dx} = 2x + 3$, with $y(1) = 2$

- Solution
  - Integration gives

$$y(x) = \int (2x + 3)\, dx = x^2 + 3x + C$$

  - For x = 1, the substitution gives

$$y(1) = 2 = 1 + 3 + C$$

  - So C = - 2, and finally

$$y(x) = x^2 + 3x - 2$$

The definite integral

- The ***definite integral*** is defined by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

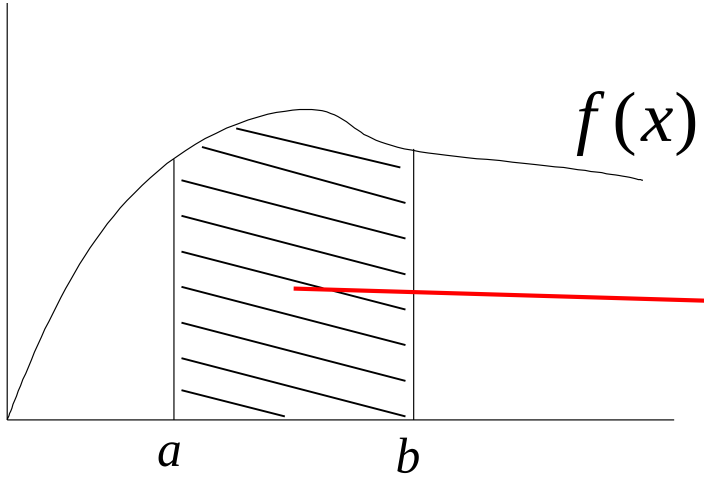
- Example

shorthand

$$\int_1^2 2x\,dx = \boxed{\left[x^2\right]_1^2} = 2^2 - 1^2 = 3$$

The definite integral

- Application: The definite integral equals the area under a function



$$= \int_a^b f(x)\,dx$$

Example

- The definite integral $\int_1^2 2x\,dx = \left[x^2\right]_1^2 = 2^2 - 1^2 = 3$